THE REAL BEHAVIOUR OF COHESIONLESS GRANULAR MATERIALS UNDERGOING DEFORMATION. A REVIEW OF STRESS-STRAIN RELATIONS AND A THEORETICAL APPROACH TO AN EXPERIMENTAL TECHNIQUE*

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The paper deals with the real behaviour of granular materials undergoing deformation. A brief review of stress-strain relations in the mechanics of particulate solids is presented. A theoretical approach is described to an experimental technique of observation of the stress and strain in a granular material in an advanced stage of deformation. The boundary condition and the stress field of Prandtl's solution at limiting equilibrium for a wedge of granular solid is described. A theoretical solution of the velocity field is given using the model due to Josselin de Jong.

Description of the flow behaviour of granular solids in storage, transport or processing units is extremely difficult. A characteristic feature of all these operations are large and relatively rapid deformations.

To date there are relatively few papers in the literature on the mechanics of granular solids dealing with the distribution of the stress and strain in a flowing granular material. The majority of papers are based on the theory of soil mechanics but there is a great deal of inconsistency in the utilisation of these models when applied to flowing granular solids and the agreement between theory and experiment is also far from being satisfactory. The aim of this work is to develop a suitable theoretical basis for selecting a proper experimental technique for the determination of the stress and strain field in a flowing granular material undergoing relatively large and rapid deformations. A part of this study is therefore devoted to a literature survey of the stress-strain relations and particularly those which have already found use in the mechanics of particulate solids and of the recent models of soil mechanics. In the paper a new approach is also described as to the choice of a suitable experimental technique for observing distribution of the stress and strain under conditions of advanced deformation. Such advanced deformation appears from the viewpoint of granular material storage and transport operations as most important.

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BRIEF LITERATURE SURVEY

Of all constitutive relations of soil mechanics based on the assumption of existence of a plastic potential the most widely used model is that of Drucker and Prager[1]. From the concept of the plastic potential

$$F = \bar{\alpha} I_1 + \sqrt{I_{2D}} = \text{const} \tag{1}$$

associated with the surface of plasticity and from the condition of normality of the increment of plastic deformation

$$\dot{\varepsilon}_{ij} = \lambda \frac{\partial F}{\partial \sigma_{ij}} \tag{2}$$

we have the following relation for the volume behaviour of a strained granular material

$$\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} + \dot{\varepsilon}_{33} = 3\lambda\bar{\alpha}\,. \tag{3}$$

This indicates a constant increment of the volume during plastic deformation. For a better agreement of the theoretical model with the real behaviour the same author proposed[2] a concept of strengthening of the material during plastic deformation as a change of the shape or shift of the surface of plasticity.

Since then, a series of models based on the existence of the plastic potential have been presented[3-9] in the field of soil mechanics. These papers examine the problem of a suitable form of the plastic potential, solution at regular and singular points of the surface of plasticity and the application to various concrete cases, particularly those of soil mechanics.

As a most realistic theory based on the assumption of the plastic potential appears to be the "critical state concept" which is being developed since 1958 at Cambridge[10-13]. The shape of the limiting surface of state is shown schematically in Fig. 1. The spatial curve DBH is called the critical state line. If the state of a sample under investigation is characterized by a point on this curve it is strained under constant volume E without any change of the stresses N and Q. This model has been originally intended for cohesive materials (clays) but has been extended also to granular solids[14].

Models developed directly for granular materials have been based on the idea of sliding of solid grains or their assemblies. The recent theory of dilatant behaviour worked out by Rowe[15] may be put forth as a typical representative.

However, none of these recent and relatively sound models has found a broader application in the mechanics of particulate solids. The same applies to an earlier approach – the theory of marginal energy correction[16] – providing similar results.

This theory served as a starting point to derive relations for the plastic potential for the case of plane deformation[17,18].

Similarly as in the case of the theories assuming existence of the plastic potential, only earlier and less justified consitutive relations, which utilize assumptions about the volume behaviour of granular materials and mutual direction of the principal axes of the stress and strain tensors, have found application in the mechanics of particulate solids.

Of these relations it is the model proposed for granular solids by Ishlinskii[19] who assumes incompressibility of the granular material

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{4}$$

and coincidence of the directions of the principal axes of the stress and strain tensors leading to the Lévy–Mises relations

$$\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right) \Big/ \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) = \frac{\sigma_x - \sigma_y}{2\tau_{xy}} . \tag{5}$$

Another, more frequently used model of the relation between the stress and strain is the theory of Geniev[20] starting from the concept of existence of the so called

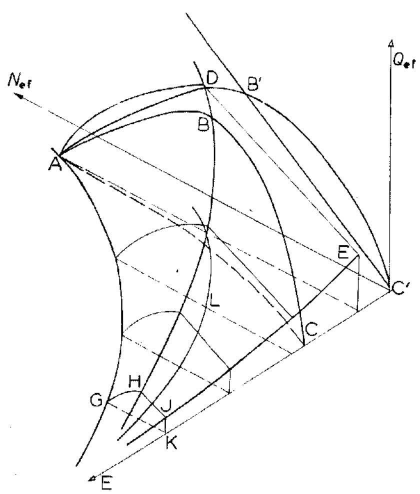

FIG. 1

Critical State Surface according to the Cambridge Theory

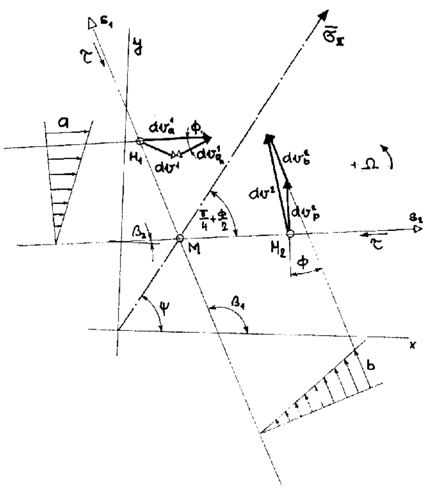

FIG. 2

Decomposition of Velocity according to de Josselin de Jong's Model active sliding characteristic line. Eq. (5) is then replaced by the equation

$$\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \left[\frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \pm \frac{\partial v_x}{\partial x}\,\mathrm{tg}\,\phi\right] \Big/ \left[\frac{\partial v_x}{\partial x} \pm \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\mathrm{tg}\,\phi\right] \quad (6)$$

and supplemented by the condition of incompressibility of the granular material. The sign in Eq. (6) depends on which of the systems of the characteristic lines is taken as active. The ideal of an active sliding line corresponds to the deviation of the direction of the principal stresses and strains by a half of the angle of internal friction of the material, *i.e.* $\xi = \pm\phi/2$.

A more recent deviation theory, making use of a general deviation angle ξ[21,22], or models starting from the concept of simultaneous sliding along both systems of the sliding characteristics[23,24] have found so far practically no application in the mechanics of granular solids. A similar situation exists as far as the model of Josselin de Jong[25,26] is concerned. The decomposition of the velocities in the vicinity of the examined point M according to this model is shown in Fig. 2.

The total increase of the velocity dv_1 at the point M_1 or dv_2 at M_2 is decomposed into a component parallel to the sliding characteristic of the second system, dv_{1a} and dv_{2b}, which represent double sliding and perpendicular components, dv_{1q} and dv_{2p}, characterizing rotation of the system of the sliding lines. For the two components we can write

$$\begin{aligned} dv_{1q} \sin\phi &= dv_{1x} \sin\beta_2 - dv_{1y} \cos\beta_2\,, \\ dv_{2p} \sin\phi &= dv_{2x} \sin\beta_1 - dv_{2y} \cos\beta_1\,, \\ dv_{1a} \sin\phi &= -dv_{1x} \cos\beta_1 - dv_{1y} \sin\beta_1\,, \\ dv_{2b} \sin\phi &= -dv_{2x} \cos\beta_2 - dv_{2y} \sin\beta_2\,. \end{aligned} \quad (7)$$

Introducing quantities a and b, characterizing double sliding for which it follows from their physical meaning of frictional forces and also from the requirement of a non-negative value of dissipated energy that $a \geqq 0$ and $b \geqq 0$, we obtain after substitution and some arrangements de Jong's constitutive inequalities

$$\begin{aligned} &\left[-\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right)\cos 2\psi - \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\sin 2\psi\right]\sin\phi \leqq \\ \leqq &\left[-\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right)\sin 2\psi + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\cos 2\psi\right]\cos\phi \leqq \\ \leqq &\left[+\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right)\cos 2\psi + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\sin 2\psi\right]\sin\phi\,. \end{aligned} \quad (8)$$

These nonequalities expressed on Mohr's circle of deformation represent deviations of the directions of the principal stresses and strains in the limits

$$-(\phi/2) \leqq \xi \leqq +(\phi/2)\,. \qquad (9)$$

The constitutive inequalities are supplemented by the condition of incompressibility of the granular material, Eq. (*4*). This condition is obtained by substituting angular velocity, Ω, of the rotation of both systems of the characteristics into Eq. (*7*) and also after some arrangements aimed at elimination of the unknown angular velocity of rotation.

The above review indicates that numerous theoretical relations have been formulated in the literature on soil mechanics which more or less accurately describe the real behaviour of deformed soil. However, in the mechanics of particulate solids, where the flow in various technological units represents as a rule a very complex problem, there exists a considerable degree of uncertainty stemming from inconsistency of application of these models. To give an example, Jenike[27,28] has used Drucker's model to solve the problem of the flow of a granular material in a bunker while in another paper[29] the same author has made use of the expression for incompresibility and coaxiality (Ishlinskii); Geniev[20] has solved the same problem using his own model of the active sliding characteristic. Yet, in all these solutions the assumption regarding the properties of the materials have been the same.

The flow of granular solids in storage, transport and process equipment is usually of a very complicated nature, a number of sliding surfaces appear and numerous other factors, such as *e.g.* the geometry of the equipment, the coefficient of external friction *etc.*, play a role in the process. A whole series of phenomena have been observed, such as for example pulsation of stresses *etc.*, which cannot be so far satisfactorily accounted for.

Thus it can be concluded that the problem of stress-strain relations in an advanced stage of deformation deserves further attention. This situation was therefore analyzed and after evaluating several cases of approach it was decided that performing experiments in an apparatus satisfying the boundary conditions of the so called Prandtl's solution would be an advantageous experimental technique.

The Boundary Conditions and the Field of Stresses of Prandtl's Solution

Prandtl's theoretical solution has been based on the usual assumptions as to the behaviour of granular material. The material is regarded to be a homogeneous, isotropic continuum in the stage of limiting plastic equilibrium as shown in Fig. 3. The limiting state of the stress is governed by the straight line envelope of the Mohr's circles with an angle of internal friction ϕ and zero cohesion.

The region of the granular material confined by the arms of the acute angle AOB is subjected to an outer total stress, q_a, making an angle ϕ with the normal to OA as shown in Fig. 4. The components of the stress σ and τ on OA are then constrained by

$$\tau_a = \sigma_a \,.\, \mathrm{tg}\, \phi \,. \tag{10}$$

The arm OA is then the locus of the limiting state of stress of the granular solid having the angle of internal friction ϕ. OA is thus one of the sliding lines.

The same can be said about OB and if the directions of the tangential components of the stress, τ_a and τ_b, are identical to the directions shown in the figure then, as shown by Prandtl[30-32], both of these sliding lines belong to the same system of sliding lines formed by a bunch of straight lines intersecting at 0. The point 0 is a singular point of Prandtl's solutions.

The condition of constant angle of intersection of the sliding lines of both systems, $\pi/2 - \phi$, yields the form of the second system of the sliding lines. The second system is therefore formed by a family of logarithmic spirals and the whole system of sliding lines is thus described by

$$s_1 \equiv \Theta = \text{konst}\,,$$

$$s_2 \equiv r = r_0 \exp\left(\Theta\, \mathrm{tg}\, \phi\right) \tag{11}$$

and the form of the system of sliding lines is independent of the angle α of Prandtl's wedge.

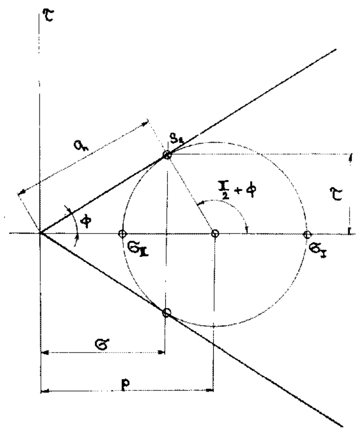

FIG. 3

Limiting State of Equilibrium of Ideal Granular Material

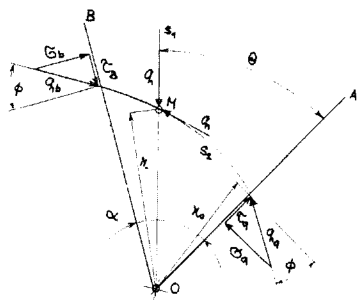

FIG. 4

Boundary Conditions of Prandtl's Wedge and Shape of Sliding Lines

Neglecting the weight of the granular solids we can write from a momentum balance about the point 0 for the total stress on the sliding line at the point M

$$q = q_a \exp\left(-2\Theta \operatorname{tg} \phi\right). \tag{12}$$

Analysis of the limiting stress of the granular material (Fig. 3) yields for individual components of the stress

$$\begin{aligned} \sigma_r &= q_a[(1 + \sin^2 \phi)/\cos \phi)] \exp\left(-2\Theta \operatorname{tg} \phi\right), \\ \sigma_\Theta &= q_a \cos \phi \,.\, \exp\left(-2\Theta \operatorname{tg} \phi\right), \\ \tau_{r\Theta} &= q_a \sin \phi \,.\, \exp\left(-2\Theta \operatorname{tg} \phi\right). \end{aligned} \tag{13}$$

The presented solution satisfies the general equations of plane limiting equilibrium for a weightless material

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\Theta}}{\partial \Theta} + \frac{\sigma_r - \sigma_\Theta}{r} = 0\,, \qquad \frac{\partial \tau_{r\Theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\Theta}{\partial \Theta} + 2\,\frac{\tau_{r\Theta}}{r} = 0\,, \tag{14}$$

since differentiation of Eqs (*13*) and substitution into Eq. (*14*) yields identity.

Application of the Model Due to de Josselin de Jong to Prandtl's Wedge

The model of de Josselin de Jong[26] yields a set of inequalities (*8*) representing the condition of deviation of the directions of the principal stresses and deformations in the range given by inequality (*9*) supplemented with the condition (*4*).

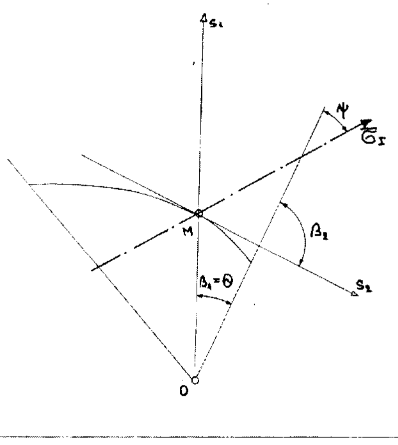

FIG. 5

Directional Angles of Sliding Lines and Trajectories of Principal Stresses in Prandtl's Wedge

Applying this model to the problem of deformation of a granular material in a wedge satisfying the boundary conditions of Prandtl's solution it is convenient to work in polar coordinates. Individual components of velocity are then related by

$$v_x = v_r \cos \Theta - v_\Theta \sin \Theta \,, \quad v_y = v_r \sin \Theta + v_\Theta \cos \Theta \,. \tag{15}$$

After some rearrangement the constitutive inequalities (8) take the form

$$\left[-2 \frac{\partial v_r}{\partial r} \cos (2\psi - 2\Theta) - \left(\frac{\partial v_\Theta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial \Theta} - \frac{v_\Theta}{r}\right) \sin (2\psi - 2\Theta)\right] \sin \phi \leqq$$
$$\leqq \left[-2 \frac{\partial v_r}{\partial r} \sin (2\psi - 2\Theta) + \left(\frac{\partial v_\Theta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial \Theta} - \frac{v_\Theta}{r}\right) \cos (2\psi - 2\Theta)\right] \cos \phi \leqq$$
$$\leqq \left[+2 \frac{\partial v_r}{\partial r} \cos (2\psi - 2\Theta) + \left(\frac{\partial v_\Theta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial \Theta} - \frac{v_\Theta}{r}\right) \sin (2\psi - 2\Theta)\right] \sin \phi \tag{16}$$

and the condition of incompressibility in Eq. (4) the form

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\Theta}{\partial \Theta} + \frac{v_r}{r} = 0 \,. \tag{17}$$

Fig. 5 shows the characteristic directional angles of the sliding lines and the trajectories of the principal stress as they follow from Prandtl's solution.

In order that we may retain a positive orientation of the sliding lines in accord with the derivation of de Josselin de Jong we write for the directional angles that

$$\beta_1 = \Theta \,, \quad \beta_2 = \Theta - (\pi/2 + \phi) \,, \quad \psi = \Theta - (\pi/4 + \phi/2) \,. \tag{18}$$

From here it then follows

$$\cos (2\psi - 2\Theta) = -\sin \phi \,, \quad \sin (2\psi - 2\Theta) = -\cos \phi \,. \tag{19}$$

Substituting Eq. (19) into (16) a relation for de Jong's model applied to Prandtl's solution results which after some arrangement takes the form

$$0 \geqq 2 \frac{\partial v_r}{\partial r} \geqq \left(\frac{\partial v_\Theta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial \Theta} - \frac{v_\Theta}{r}\right) \operatorname{tg} 2\phi \,. \tag{20}$$

This constitutive inequality is supplemented with the condition of incompressibility, Eq. (17), without change.

Experimental apparatus which would satisfy the boundary conditions of Prandtl's solution would be suitable for investigating the real behaviour of strained granular materials. Such an equipment would permit considerable deformations allowing measurement of the stress and strain. The advantage of Prandtl's solution is that the family of sliding lines is independent of the angle of the wedge and hence constant throughout the deformation process.

At odds with the real experimental conditions appear to be the neglected effects of the weight of the granular material in the theoretical solution as well as a precise simulation of the boundary conditions at the singular point, *i.e.* in the apex of the wedge. The weight of the granular solids will no doubt somewhat distort the shape of sliding lines. However, in view of the possibility of simulating precisely the shape of the boundary sliding lines and the symmetric orientation of the wedge with respect to gravity the distortion need not have any major effect. The effect of singularity in the neighborhood of the apex of the Prandtl's wedge may be eliminated by dropping this region from data processing.

LIST OF SYMBOLS

a, b quantities characterizing shear along the sliding line
E porosity of granular material
F function, surface of plasticity
I_1 first invariant of the stress tensor
I_{20} second invariant of the stress deviator
N_{ef} effective normal stress
p mean stress in granular material
q total stress on sliding surface
Q_{ef} effective shear stress
r polar radius
v velocity
x, y cartesian coordinates
$\bar{\alpha}$ coefficient
α apex angle of wedge
β direction of sliding line
ε plastic deformation
λ parameter
ξ deviation angle
σ normal stress
τ shear stress
ϕ angle of internal friction of granular material
ψ direction of principal stress
Ω angular velocity of rotation of system of sliding lines
Θ radial coordinate

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